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Spectra of some Operations on Graphs

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ABSTRACT: In this paper, we consider a finite undirected and connected simple graph $G(E, V)$ with vertex set $V(G)$ and edge set $E(G)$.We introduced a new computes the spectra of some operations on simple graphs [union of disjoint graphs, join of graphs, Cartesian product of graphs, strong Cartesian product of graphs, direct product of graphs].

Keywords: adjacency; laplacian; union of graph; join of graph; Cartesian product, strong Cartesian product; direct product.

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INTRODUCTION

Graph theory and its applications has a long history, in structural mechanics and in particular

nodal ordering and graph partitioning are well documented in the literature, Kaveh [11-12]. Algebraic graph theory can be considered as a branch of graph theory, where eigenvalues and eigenvectors of certain matrices are employed to deduce the principal properties of a graph. In fact eigenvalues are closely related to most of the invariants of a graph, linking one extremal property to another. These eigenvalues play a central role in our fundamental understanding of graphs. Most of the definitions on algebraic graph theory in the present interesting books such as Biggs [2], Cvetković et al. [5], and Godsil and Royle [10]. One of the major contributions in algebraic graph theory is due to Fiedler [9], where the properties of the second eigenvalue and eigenvector of the Laplacian of a graph have been introduced. This eigenvector, known as the Fiedler vector is used in graph nodal ordering and bipartition, Refs. [14-17].

The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. The related matrix - the adjacency matrix of a graph and its eigenvalues were much more investigated in the past than the Laplacian matrix. In the same time, the Laplacian spectrum is much more natural and more important than the adjacency matrix spectrum because of it numerous application in mathematical physics, chemistry and financial mathematics (see papers [1, 3, 4, 6, 7, 8]).

•The adjacency matrix, $A = A(G) = (a_{ij})$ of Gis an $n \times n$ symmetric matrix,G(finite undirected and connected simple graph)

$$a_{ij} = \begin{cases} 1, & \text{if} \quad v_i v_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

•The Laplacian matrix of G is the matrix $L = L(G) = l_{ij} = D - A$,

$$l_{ij} = \begin{cases} d_i, & \text{if} \quad i = j, \\ -1, & \text{if} \quad v_i v_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Where D is a diagonal degree matrix $\left(D = (d_1, d_2, \dots, d_n)\right)$ of a graph G, d_i is the degree of vertex i.

•The characteristic polynomial of $A(G)$ or G is defined as $P(G,\lambda) = det\left(\lambda I - A(G)\right)$.

• The roots of $P(G,\lambda)$ are the eigenvalues of $A(G)$. We will call them also the eigenvalues of G.

•The (ordinary) spectrum of a finite graph G is by definition the spectrum of theadjacency matrix $A(G)$, that is, its set of eigenvalues together with their multiplicities.

1. Some operations on graphs and spectra

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. For the other operations, we assume that G and H are graphs with disjoint vertex-sets, $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$:

(1) The union $G \cup H$ has vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

(2) The join $G + H$ is obtained from $G \cup H$ by adding all of the edges from vertices in G to those in H.

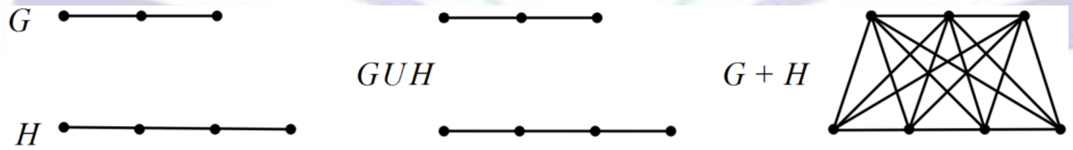

Figure 1

Theorem 2.1, Let G be the union of disjoint graphs $G_1, G_2, G_3, \dots G_n$; $i.e.$ $(G = \cup_{i=1}^{n} G_i)$.

Then$P(G,\lambda) = \prod_{i=1}^{n} P(G_i,\lambda)$

Proof. For any square matrices $A_1, A_2, A_3, \dots A_n$ not necessarily of the same order.The claim follows at once from the relation

$$\det\begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_n \end{bmatrix} = \det(A_1)\det(A_2)\det(A_3)\cdots\det(A_n).$$

Theorem Kel'mans [13] 2.2, Let $G + H$ denote the join of G and H, then

$$\boldsymbol{P}(G+H,\lambda) = \frac{\lambda(\lambda-n_1-n_2)}{(\lambda-n_1)(\lambda-n_2)}\boldsymbol{P}(G,\lambda-n_2)\boldsymbol{P}(H,\lambda-n_1)$$

Where n_1 and n_2 are orders of G and H, respectively $\boldsymbol{P}(G,\lambda)$ is the characteristic polynomial of the Laplacian matrix of G.

•The wheel graph, W_{n+1} the graph which is given by $W_{n+1} = K_1 + C_n$, where C_n is the cycle graph with n vertices and K_1 is any new vertex.

•The fan graph, F_{n+1} the graph which is given by $F_{n+1} = K_1 + P_n$, where P_n is the path graph with n vertices and K_1 is any new vertex.

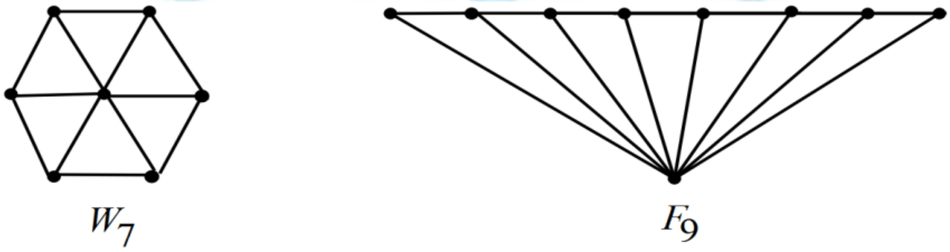

Figure 2

Note that: Chebyshev polynomials**:**

(1) Chebyshev polynomial of the first kind $T_n(x) = \cos(n\cos^{-1}x)$.

(2) Chebyshev polynomial of the second kind $U_n(x) = \sin(n\cos^{-1}x)$

Theorem 2.3, (1) Laplacian spectrum of the fan graph F_{n+1} are

$$\{0, n+1, 3-2\cos\frac{\pi i}{n}, i = 1, \ldots, n-1\}$$

(2) Laplacian spectrum of the wheel graph W_{n+1} are

$$\left\{0, n+1, 3-2\cos\frac{2\pi i}{n}, i = 1, \ldots, n-1\right\}.$$

Proof. (1) Since Laplacian polynomial of the path graph $\boldsymbol{P}(P_n,\lambda) = \lambda U_{n-1}(\frac{\lambda-2}{2})$ and $\boldsymbol{P}(K_1,\lambda) = \lambda$, then by theorem (Kel'mans)the Laplacian polynomial of the fan graph F_{n+1} is given by the formula

$$\boldsymbol{P}(F_{n+1},\lambda) = \lambda(\lambda-n-1)U_{n-1}(\tfrac{\lambda-3}{2}).$$

Thus Laplacian spectrum F_{n+1} are $\left\{0, n+1, 3-2\cos\frac{\pi i}{n}, i = 1, \ldots, n-1\right\}$

(2) Since Laplacian polynomial of the cycle graph $\boldsymbol{P}(C_n,\lambda) = 2\lambda[T_{n-1}\left(\frac{\lambda-2}{2}\right)-1]$ and $\boldsymbol{P}(K_1,\lambda) = \lambda$, then by theorem (Kel'mans) the Laplacian polynomial of the wheel graph W_{n+1} is given by the formula $\boldsymbol{P}(W_{n+1},\lambda) = 2\lambda(\lambda-n-1)[T_{n-1}\left(\frac{\lambda-3}{2}\right)-1]$.

Thus Laplacian spectrum W_{n+1} are $\left\{0, n+1, 3-2\cos\frac{2\pi i}{n}, i = 1, \ldots, n-1\right\}$. □

Example 2.4: The Laplacian characteristic polynomial of the wheel graph W_7 is

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$$\boldsymbol{P}(W_7,\lambda) = det\left(\lambda I - L(W_7)\right) = det\begin{bmatrix} \lambda-6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \lambda-3 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & \lambda-3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & \lambda-3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & \lambda-3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & \lambda-3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & \lambda-3 \end{bmatrix}$$

$= \lambda^7 - 24\lambda^6 + 231\lambda^5 - 1140\lambda^4 + 3036\lambda^3 - 4128\lambda^2 + 2240\lambda.$

Thus Laplacian spectrum W_7 are $\{0, 7, 2, 4, 5, 4, 2\}$ which us (2) in Theorem 2.3, Laplacian spectrum W_7 are $\left\{0, 7, 3 - 2\cos\frac{2\pi i}{6}, i = 1, \ldots, 5\right\}$.

(2) The Laplacian characteristic polynomial of thefan graph F_9 is

$$\boldsymbol{P}(F_9,\lambda) = det\left(\lambda I - L(F_9)\right) = det\begin{bmatrix} \lambda-8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \lambda-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \lambda-3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \lambda-3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \lambda-3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & \lambda-3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & \lambda-3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda-3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda-2 \end{bmatrix} = \lambda^9 -$$

$$30\lambda^8 + 372\lambda^7 - 2502\lambda^6 + 10000\lambda^5 - 24318\lambda^4 + 35096\lambda^3 - 27438\lambda^2 + 8883\lambda.$$

Thus Laplacian spectrum F_9 are $\{0, 1.1522, 1.5858, 2.2346, 3, 3.7654, 4.4142, 4.8478, 9\}$,

which us (1) in Theorem 2.3, Laplacian spectrum F_9 are $\left\{0, 9, 3 - 2\cos\frac{\pi i}{8}, i = 1, \ldots, 7\right\}$. □

Example 2.5 : (1) Find Laplacian spectrum of the complete graph on n vertices $\mathrm{K_n}$.

(2) Find Laplacian spectrum of the complete bipartite graph $\mathrm{K_{m,n}}$.

Solution: (1) We want to show that $\boldsymbol{P}(\mathrm{K_n},\lambda) = \lambda(\lambda - n)^{n-1}$. To solve this problem we use induction by number of vertices n. For, $\mathrm{K_1}$ is a singular vertex. Its Laplacian matrix

$L(\mathrm{K_n}) = \{0\}$. Hence $\boldsymbol{P}(\mathrm{K_1},\lambda) = \lambda$.

Hence the statement is true for $n = 1$. Suppose that for given n the equality $\boldsymbol{P}(\mathrm{K_n},\lambda) = \lambda(\lambda - n)^{n-1}$ is already proved. It is easy to see that

$\mathrm{K_{n+1}}$ is a join of $\mathrm{K_n}$ and $\mathrm{K_1}$. By Kel'mans theorem we get

$$\boldsymbol{P}(\mathrm{K_n},\lambda) = \frac{\lambda(\lambda-n-1)}{(\lambda-1)(\lambda-n)}\boldsymbol{P}(\mathrm{K_1},\lambda-n)\boldsymbol{P}(\mathrm{K_n},\lambda-1)$$

$$= \frac{\lambda(\lambda-n-1)}{(\lambda-1)(\lambda-n)}(\lambda-n)(\lambda-1)(\lambda-1-n)^{n-1} = \lambda(\lambda-n-1)^{n}$$

Hence, the Laplacian spectrum of $\mathrm{K_n} = \{0,\ \mathrm{n^{n-1}}\}$.

(2) Let us note that $\mathrm{K_{m,n}}$ is a join of x_m and x_n, where x_k is a disjoint union of k vertices. We have $L(x_k) = D(x_k) - A(x_k) = 0_k - 0_k = 0_k$, where 0_k is $k \times k$ zero matrix.

Hence, $\boldsymbol{P}(x_k,\lambda) = \lambda^k$. By Kel'mans theorem we obtain

$$\boldsymbol{P}(\mathrm{K_{m,n}},\lambda) = \frac{\lambda(\lambda-m-n)}{(\lambda-n)(\lambda-m)}\boldsymbol{P}(x_n,\lambda-m)\boldsymbol{P}(x_m,\lambda-n)$$

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$$= \frac{\lambda(\lambda - m - n)}{(\lambda - n)(\lambda - m)} (\lambda - m)^n (\lambda - n)^m$$

$$= \lambda(\lambda - m - n)(\lambda - m)^{n-1}(\lambda - n)^{m-1}$$

Hence, the Laplacian spectrum of $K_{m,n}$ is $\{0, m^{n-1}, n^{m-1}, m + n\}$. □

3. Types of graph products and spectra

(1) Cartesian product $G \times H or G \square H$ has the vertex-set $V(G) \times V(H)$, and

(u_i, v_j) is adjacent to (u_h, v_k) if either:

(a) u_i is adjacent to u_h in G and $v_j = v_k$,or

(b) $u_i = u_h$ and v_j is adjacent to v_k in H.

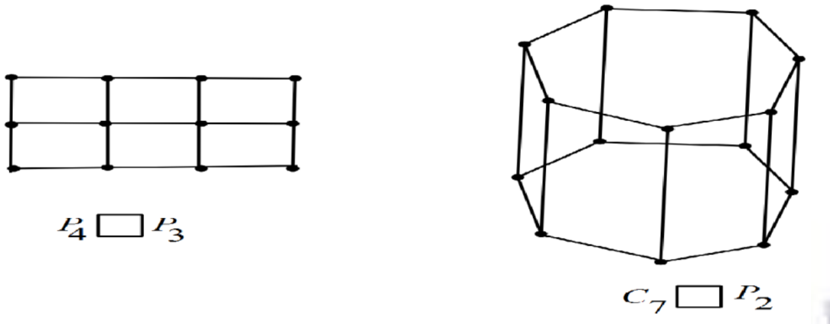

Figure 3

(2) Strong Cartesian product $G \boxtimes H$ has the vertex-set $V(G) \times V(H)$, and

(u_i, v_j) is adjacent to (u_h, v_k) if:

(a) u_i is adjacent to u_h in G and $v_j = v_k$,

(b) $u_i = u_h$ and v_j is adjacent to v_k in H,

(c) u_i is adjacent to u_h in G and v_j is adjacent to v_k in H; i.e. $(u_i u_h \in G \text{ and } v_j v_k \in \mathrm{H})$.

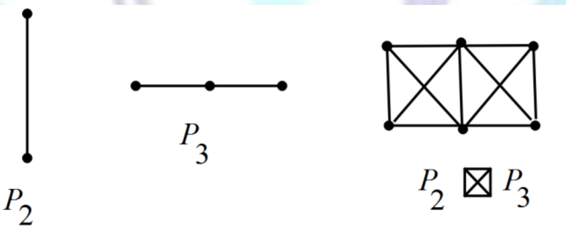

Figure 4

(3) Direct (Kronecker) product $G \otimes H \text{ or } G * H$ has the vertex-set $V(G) \times V(H)$, and

(u_i, v_j) is adjacent to (u_h, v_k) if: $u_i u_h \in G \text{ and } v_j v_k \in \mathrm{H}$.

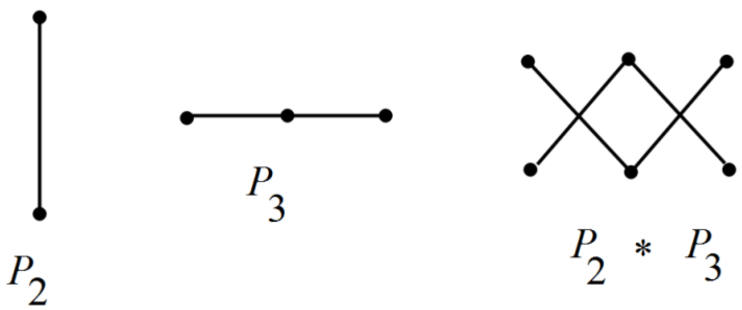

Figure 5

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Remark [16]: (i) The cylinder graph can be written on the image $P_m \times C_n$.

(ii) The adjacency matrix of $G \times H$ can be written as $A(G) \otimes I_m + I_n \otimes A(H)$.

Here $\otimes$ is tensor (kronecker) product of matrices.

(iii) The adjacency matrix of $G \otimes H$can be written as $A(G) \otimes A(H)$.

(iv) The adjacency matrix of $G \boxtimes H$ can be written as $\big((A(G)+I) \otimes (A(H)+I)\big) - I$.

(v) The Kronecker product of two matrices A and B, is the matrix we get by replacing the ij − th entry of A by aijB, for all i and j.

As an example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

(vi) The Kronecker product has the property that if **B**, **C**, **D**, and ***E*** are four matrices, such that **BD** and ***CE*** exists, then: $(\mathbf{B} \otimes \mathbf{C})(\mathbf{D} \otimes \mathbf{E}) = \mathbf{BD} \otimes \mathbf{CE}$.

Theorem 3.1. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of G and $\square_1, \square_2, \ldots, \square_m$ are eigenvalues of H.Then

(1) the eigenvalues of $G \times H$are $\lambda_i + \square_j$, for $1 \leq i \leq n$ and $1 \leq j \leq m$.

(2) the eigenvalues of$G \otimes H$ are $\lambda_i \square_j$, for $1 \leq i \leq n$ and $1 \leq j \leq m$.

(3) the eigenvalues of $G \boxtimes H$ are $(\lambda_i + 1)\big(\square_j + 1\big) - 1$ or $\lambda_i \square_j + \lambda_i + \square_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Proof. (1)Let A and B be the adjacency matrix of G and Hrespectively. For any eigenvalue λ and x eigenvector of A, any eigenvalue $\square$ and $y$eigenvector of B. We have $Ax = \lambda x$ and $By = \square y$. It follows that

$$(A \otimes I_m + I_n \otimes B)(x \otimes y) = (A \otimes I_m)(x \otimes y) + (I_n \otimes B)(x \otimes y)$$

$$= Ax \otimes I_m y + I_n x \otimes By = \lambda x \otimes y + x \otimes \square y$$

$$= \lambda(x \otimes y) + \square(x \otimes y) = (\lambda + \square)(x \otimes y).$$

Thus, $\lambda + \square$ is an eigenvalue of $G \times H$.
(2) $(A \otimes B)(x \otimes y) = Ax \otimes By = \lambda x \otimes \square y = \lambda\square(x \otimes y)$.

Thus, $\lambda \square$ is an eigenvalue of $G \otimes H$.

(3) $\Big(\big((A+I) \otimes (B+I)\big) - I\Big)(x \otimes y) = \big((A+I) \otimes (B+I)\big)(x \otimes y) - (x \otimes y)$

$$= (A+I)x \otimes (B+I)y - (x \otimes y)$$

$$= (Ax + x) \otimes (By + y) - (x \otimes y)$$

$$= (\lambda x + x) \otimes (\square y + y) - (x \otimes y)$$

$$= (\lambda + 1)x \otimes (\square + 1)y - (x \otimes y)$$

$$= (\lambda + 1)(\square + 1)(x \otimes y) - (x \otimes y)$$

$$= ((\lambda + 1)(\square + 1) - 1)(x \otimes y).$$

Thus, $(\lambda + 1)(\square + 1) - 1$is an eigenvalue of $G \boxtimes H$. □

Example 3.2: Let the cylinder graph $G = P_2 \times C_7$, the characteristic polynomial of G is

$P(G,\lambda) = det\left(\lambda I - A(G)\right) = \lambda^{14} - 21\lambda^{12} + 154\lambda^{10} - 476\lambda^{8} - 4\lambda^{7} + 623\lambda^{6} - 56\lambda^{5} - 343\lambda^{4} + 84\lambda^{3} + 63\lambda^{2} - 28\,\lambda + 3$. Thus adjacency spectrum of the cylinder graph are $\{-2.8019, -2.8019, -1.445, -1.445, -0.8019, -0.8019, 0.247, 0.247, 0.555, 0.555, 1, 2.247, 2.247, 3\}$. By Theorem 2.3 (2), Adjacency spectrum of the cylinder graph are $\{\lambda_i + ⍰_j : i = 1, 2\ and\ j = 1, \dots, 7\}$, where $\lambda_i = \{-1, 1\}\ and\ ⍰_j = \{-1.8019, -1.8019, -0.445, -0.445, 1.247, 1.247, 2\}$.

4. CONCULSION

In this paper, we give a new computes of laplacian spectrum of some graphs which represent addition of two graphs. Also we give a new computes of adjacency spectrum of some graphs which represent product of two graphs.

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